CONTROL THEORY FUNDAMENTALS

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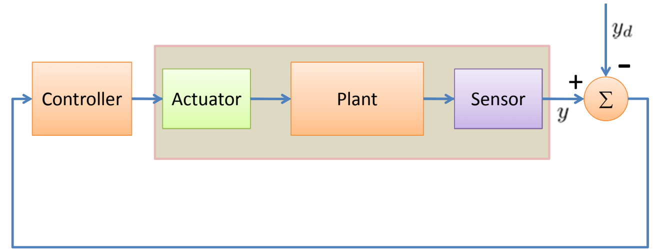
Feedback Control



- y must follow y_d with
  - » Speed
  - » Accuracy
- Feeback: *Measure – Compare - Correct*

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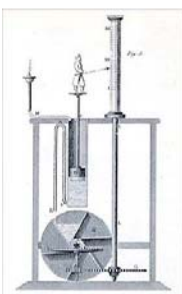
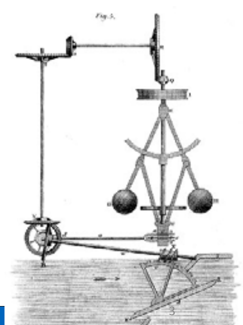
Feedback Control: A History



As the speed of the prime mover increases, the central spindle of the governor speeds up causing the two masses on lever arms to move up, pulls down a thrust bearing, closing a throttle valve.



From 3rd century BC. The hour indicator ascends as water flows in. A series of gears rotate a cylinder to correspond to the temporal hours

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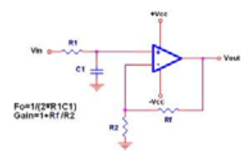
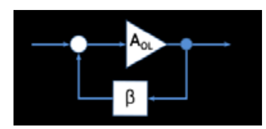
Feedback Control: A History (contd.)





- Negative feedback improves performance (gain stability, linearity, frequency response, step response) and reduces sensitivity to parameter variations due to manufacturing or environment
- Used extensively in World War II
- Launched control theory in its own right

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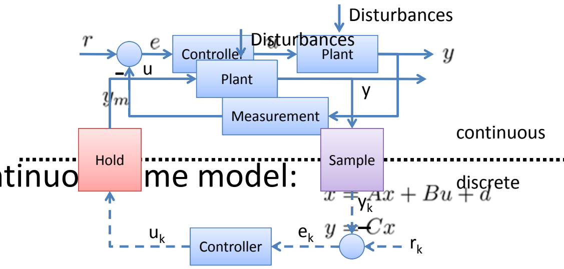
Dynamic Models

Starting point: Determination of Plant-model

• Sampled-data systems:



• Continuous-time model: $\dot{x} = Ax + Bu + d$, $y = Cx$

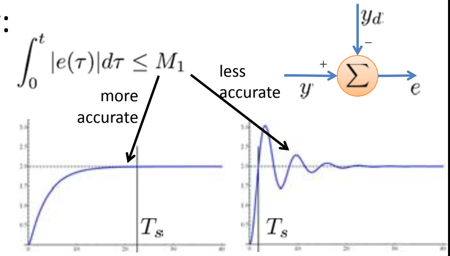
• Discrete-time model: z_k: Value of $z(t)$ at time $t = t_k$

$$y_k = a_1 y_{k-1} + a_2 y_{k-2} + \ldots + a_n y_{k-n} + b_0 u_k + \ldots + b_m u_{k-m}$$

Control Performance Metrics

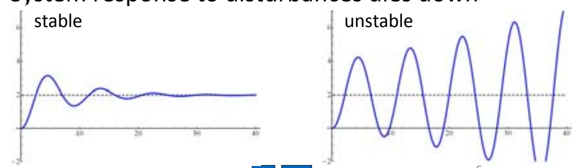
• Tracking accuracy: $\int_0^t |e(\tau)| d\tau \le M_1$

• Speed: $T_s \le T$



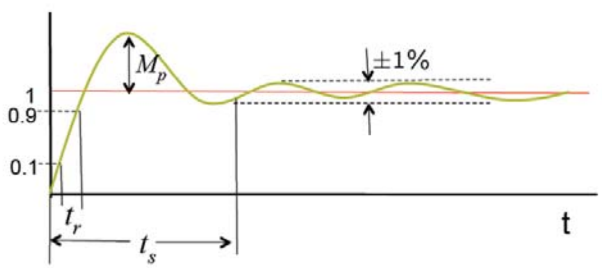
• Stability:

System response to disturbances dies down



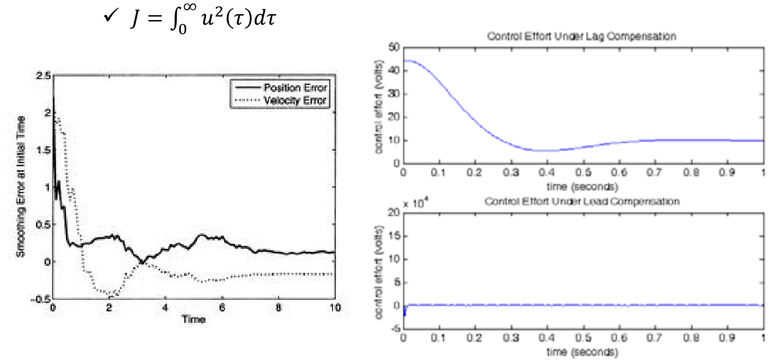
Transient Performance Metrics: Examples

• Metric 1: Settling time t_s
  ✓ Time it takes the system transients to decay
• Metric 2: Peak Overshoot M_p
  ✓ Maximum Overshoot

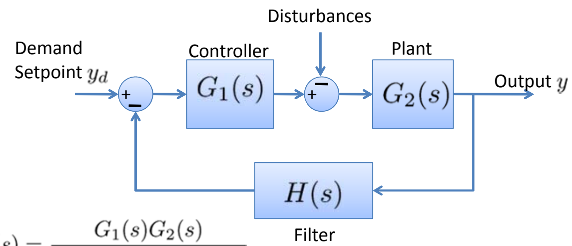


Steady-state Performance Metrics: Examples

• Metric 1: Cost of tracking error $e = y - y_d$
  ✓ $J = \int_0^\infty e^2(\tau) d\tau$
• Metric 2: Cost of Control Effort u
  ✓ $J = \int_0^\infty u^2(\tau) d\tau$



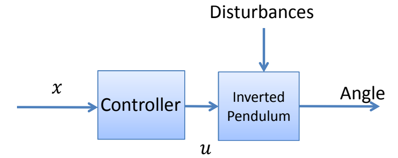
Frequency Domain Methods



$$\frac{y}{y_d} = T(s) = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

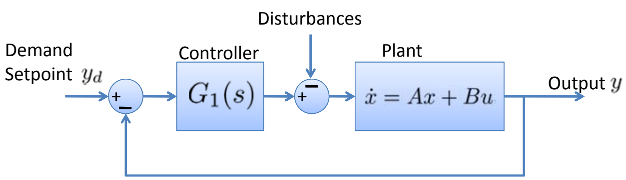
- Design $G_1(s)$ and $H(s)$ so that $y(t) - y_d(t) \to 0$
- Ensure speed and accuracy of the closed-loop by studying the forward-loop transfer function $G_1(s)G_2(s)H(s)$
- Bode-plots, Root-locus, Nyquist plots are useful tools

A typical control design



- Plant (inverted pendulum): $\ddot{x} = ax + u + d$ $\quad (a = \frac{g}{\ell})$
- Controller (horizontal force): $u = -k_1\dot{x} - k_2 x$
- Closed-loop dynamics: $\ddot{x} + k_1\dot{x} + (k_2 - a)x = d$
- Stability: Responses to an impulse in d must decay. $\Rightarrow k_1 > 0, k_2 > a$
- Transient performance: $(k_2 - a) \sim \left(\frac{1}{4}\right) k_1^2 \Rightarrow e^{-k_1 t}$
- Control Effort: Increases with k_1, k_2
- Optimal control: Trade-off between performance and control cost.

State Space Methods



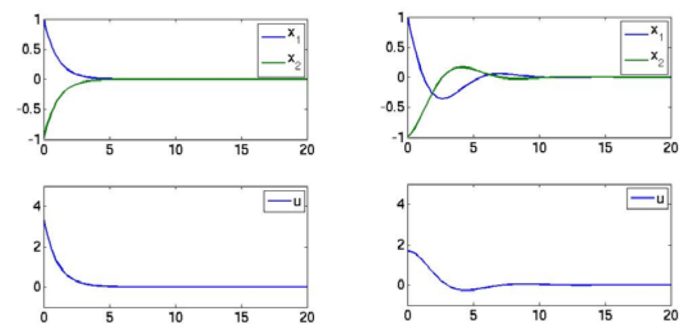
- Choose $G_1(s)$ so that
  - closed-loop system is stable (Eigenvalues of A have negative real parts – A is Hurwitz)
  - $y(t) \to y_d(t)$
  - a cost function is optimized. For example,

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt$$

leads to **Optimal Control:**
Solution is of the form $u = -Kx$ with $K = R^{-1}B^T P$
with $A^T P + PA - PBR^{-1}B^T P = -Q$

Optimal Control

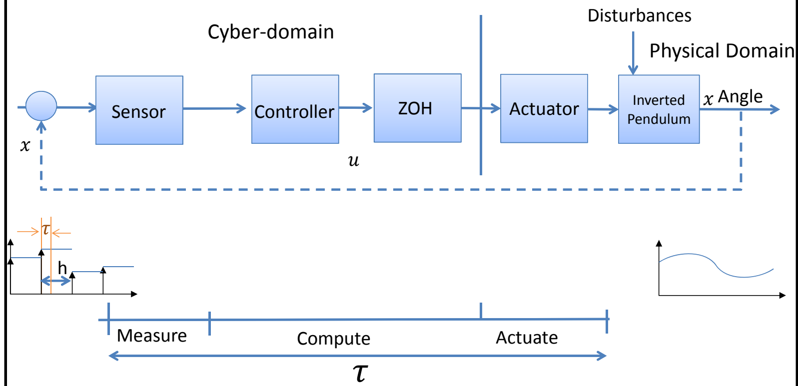
- Large k_1, k_2
- Optimal k_1, k_2



Feedback gain matrix: K = [6.4; 3]
Poles: [-1; -2]
J = 30.035

Feedback gain matrix: K = [2.08; 0.41]
Poles: [-0.42 + 0.73j; -0.42 - 0.73j]
J = 17.911

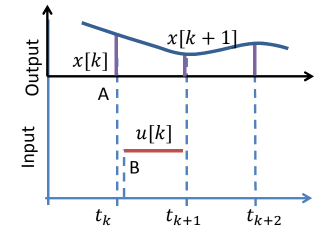
Computer-controlled Systems



- Sample every h secs.
- Nyquist sampling: The sampling frequency $\left(\frac{1}{h}\right)$ should be at least twice the highest frequency in the (physical) system dynamics.

A Standard Sampled-Data Problem

- Physical system: $\dot{x}(t) = A_c x(t) + B_c u(t)$
- Multiple both sides by $e^{-A_c t}$
  $e^{-A_c t}\dot{x}(t) - e^{-A_c t}A_c x(t) = e^{-A_c t}B_c u(t)$
  $\frac{d}{dt}\left(e^{-A_c t}x(t)\right) = e^{-A_c t}B_c u(t)$
- Integrate over $[t_k, t_{k+1}]$



$$e^{-A_c(kh+h)}x[k+1] - e^{-A_c kh}x(kh) = \int_{kh}^{kh+h} e^{-A_c \eta} B_c u(\eta)\, d\eta$$

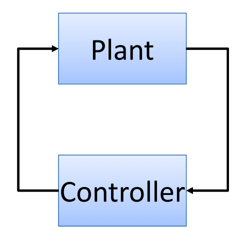
- Multiply both sides by $e^{A_c(kh+h)}$
- $u(\eta) = u[k], over\ [t_k, t_{k+1}]$
- $x[k+1] = e^{A_c h}x[k] + \int_{kh}^{kh+h} e^{-A_c(\eta - kh - h)}B_c\, d\eta \cdot u[k]$

$$\Rightarrow x[k+1] = Ax[k] + Bu[k]$$

- Assumes that $u[k]$ is available immediately after t_k

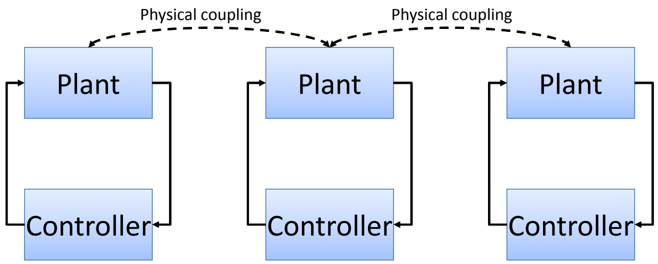
NETWORKED CONTROL SYSTEMS

Traditional Control System



- Centralized System
- Analog signal transfer
- Point-to-point communication
- One wire per signal
- Ideal signal transfer assumed

Decentralized Control System



- Decentralized System
- Physical coupling between different subsystems
- Each subsystem is controlled by a local controller
- No information exchange among controllers

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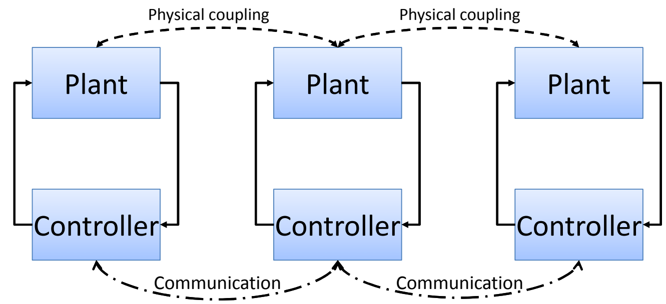
Networked Control Systems (NCS)



A Networked Control System is a spatially distributed control system where information is exchanged over a (digital) network.

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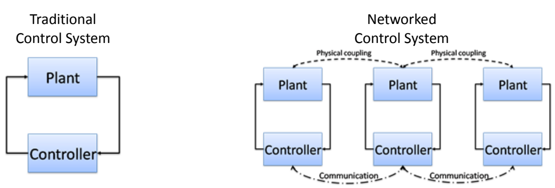
Features of NCS



- Sensors and controllers can be added or removed without wiring efforts
- Increased re-configurability
- Simplification of diagnosis procedures and maintenance
- Hence, reduction of cost
- Efficient sharing of data via network

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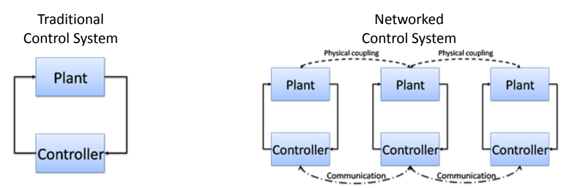
Advantages of NCS



- Reduced complexity, wiring, and cost of system
- Easy maintenance, diagnosis, and reconfiguration
- Increased flexibility and autonomy

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Applications of NCS

- Automobile industry in 1970's
  - Driven by reduced cost for cabling, modularization of systems, and flexibility in car manufacturing
- A wide range of applications at present

- Engineering Networks
  - Manufacturing automation
  - Automotive Systems
  - Aircraft
  - Teleoperation & Remote Surgery
  - Building automation
  - Automoted highway systems
  - Environmental monitoring and control
- Physical/ biological/ ecological networks
  - Synchronization networks
  - Flock of birds/school of fisch
  - Gene/cell networks
  - Food webs
  - Social networks

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Example 1: Automotive Systems

- More than 50 control units
  - Engine Control
  - Idle speed control
  - Drive by wire
  - Lights
  - Diagnosis
  - Cruise Control

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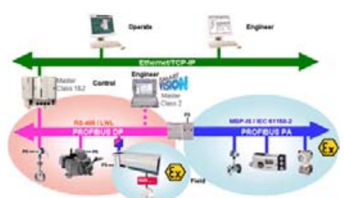
Example 2: Manufacturing

- Example of complex process control

Highly interconnected control systems in a manufacturing process



An industrial bus protocol: Profibus

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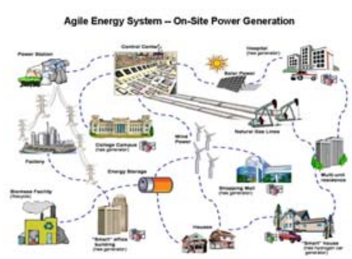
Example 3: Power Networks



- Heterogeneous power generation networks
  - Many small power plants connected:
    - Solar
    - Wind mill
    - Nuclear power plants
    - Gas turbines
    - ....

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Example 4: Traffic Management

- Motivation:
  - Increase traffic throughput
  - Avoid congestion
- Install wireless traffic sensors
- Model traffic as partial differential equations

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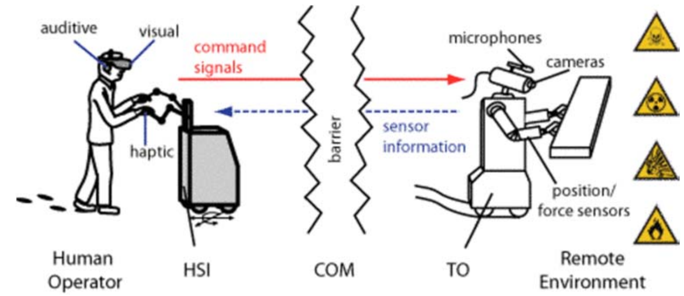
Example 5: Telepresence Systems

- Cooperative Telemanipulation



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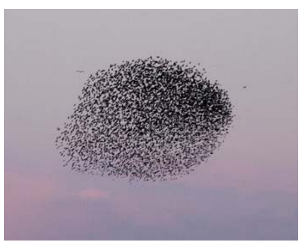
Example 6: Biological systems



- Each bird/fish adjusts its velocity and direction only according to its neighbors

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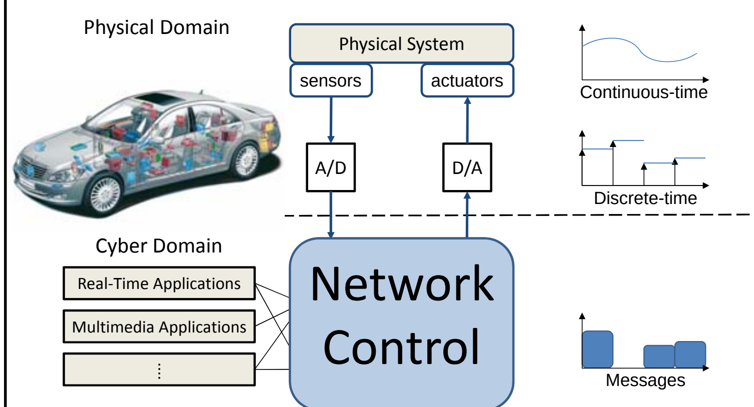
Implications of NCS



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Implications of Network Control

1. Delays and packet dropouts:
   - Non-ideal signal transmission
   - Delays and packet dropouts are the consequence
   - Delays are a source of instability and performance deterioration
   - Delay depends on network configuration, number of particpators, routing transients, aggregate flows, network topolgies
   - Transmission delays may be non-deterministic

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Implications of NCS (contd.)

2. Limited network resources
   - Multiple sensors and system communicating over a shared network
   - Network bandwidth is essential in the design of the system
   - Need for optimal scheduling and prioritizing

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NCS: Challenges (3)

3. Synchronization of local clocks
   - Clock offset may drift
   - Time and durations may differ for each component in the NCS

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Highlights of NCS Solutions

- Analysis and synthesis of controllers that are robust to
  (1) delays
  (2) varying delays
  (3) packet dropouts

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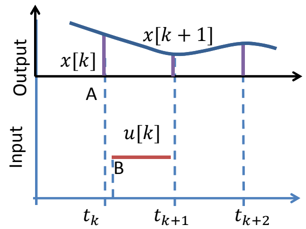
A Standard Sampled-Data Problem

- Physical system: $\dot{x}(t) = A_c x(t) + B_c u(t)$
- Multiple both sides by $e^{-A_c t}$

$$\frac{d}{dt}(e^{-A_c t}x(t)) = e^{-A_c t}B_c u(t)$$

- Integrate over $[t_k, t_{k+1}]$



$$e^{-A_c(kh+h)}x[k+1] - e^{-A_c kh}x(kh) = \int_{kh}^{kh+h} e^{-A_c \eta}B_c u(\eta)\,d\eta$$

- Multiply both sides by $e^{A_c(kh+h)}$
- $u(\eta) = u[k], over\ [t_k, t_{k+1}]$
- $x[k+1] = e^{A_c h}x[k] + \int_{kh}^{kh+h} e^{-A_c(\eta-kh-h)}B_c\,d\eta \cdot u[k]$

$$\Rightarrow x[k+1] = Ax[k] + Bu[k]$$

- Assumes that $u[k]$ is available immediately after t_k

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Effect of Network: $\tau \neq 0$

- Physical system: $\dot{x}(t) = A_c x(t) + B_c u(t)$
- With sampling, and integration over$[t_k, t_{k+1}]$

$x[k+1] = e^{A_c h}x[k] + \int_{kh}^{kh+h} e^{-A_c(\eta-kh-h)}B_c u(\eta)d\eta$
- $u(\eta) = u[k-1], \quad [t_k, t_k+\tau]$
- $u(\eta) = u[k], \quad [t_k+\tau, t_{k+1}]$

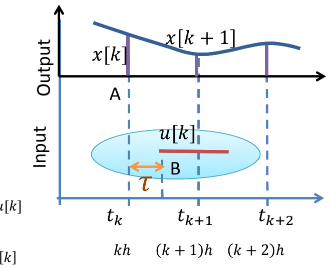
$x[k+1] = e^{A_c h}x[k] +$

$\int_{kh}^{kh+\tau} e^{-A_c(\eta-kh-h)}B_c\,d\eta \cdot u[k-1] + \int_{kh+\tau}^{kh+h} e^{-A_c(\eta-kh-h)}B_c\,d\eta \cdot u[k]$

$$= \underbrace{e^{A_c h}}_{A}x[k] + \underbrace{\int_{h-\tau}^{h} e^{A_c \nu}\,d\eta \cdot B_c}_{B_2} \cdot u[k-1] + \underbrace{\int_{0}^{h-\tau} e^{A_c \nu}\,d\eta \cdot B_c}_{B_1} \cdot u[k]$$

$$\Rightarrow x[k+1] = Ax[k] + B_1 u[k] + B_2 u[k-1]$$

- τ can be a significant fraction of h.
- Design $u[k]$ assuming $B_2 \sim 0$. Guarantee robustness



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Stability Tools

We need to accommodate different cases where $\tau \sim 0, \tau \sim \alpha h, \alpha \ll 1, \tau \sim \beta h, \beta \sim 1$.
- Switched Systems

- Determine robustness of control designs to τ.
- Definition:

A is Schur $\Rightarrow$ all eigenvalues of A are inside the unit circle

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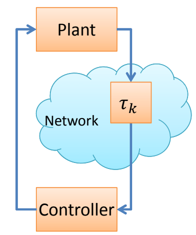
Control designs that are robust to τ

$$\dot{x} = Ax + B\hat{y}, \quad y = Cx$$

$$\hat{y}_k = y_k \quad \forall k \in \mathbb{N}$$

$$\hat{y}(t) = \begin{cases} \hat{y}_{k-1}, & t \in [t_k, t_k + \tau_k) \\ \hat{y}_k, & t \in [t_k + \tau_k, t_{k+1}) \end{cases}$$



Theorem: Assuming there exist constants $h > \tau \geq 0$ such that

$$t_{k+1} - t_k = h, \qquad \tau_k = \tau, \qquad \forall k \in N$$

the NCS in the figure above is exponentially stable if the closed-loop matrix is Schur.

Imposes limits on the delay for satisfactory behavior

(1) M. S. Branicky et al., "Stability of networked control systems: Explicit analysis of delay," Amer. Contr. Conf., 2000, vol. 4

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Control designs that are robust to variable delay[2]

$$\xi_{k+1} = \bar{A}(\tau_k)\xi_k, \ \tau_k \in [0, \tau_{\max}]$$

with

$$\bar{A}(\tau_k) = \begin{bmatrix} e^{Ah} - \Gamma_0(\tau_k)BK & -\Gamma_1 BK \\ I & 0 \end{bmatrix}$$

and $\xi_k = \begin{pmatrix} x_k^T & x_{k-1}^T \end{pmatrix}^T$

Theorem:
Given the system above with the delay-dependent matrix $\bar{A}(\tau_k)$. If there exists a solution to the discrete-time Lyapunov matrix inequalities

$$P = P^T > 0$$
$$\bar{A}^T P \bar{A} - P < 0, \forall \bar{A} \in \mathcal{A}$$

for a suitable chosen finite set of matrices $\mathcal{A}$, then the system is robustly globally asymptotically stable for any sequence of delays $\tau_k \in [0, \tau_{\max}]$

(2) M. Cloosterman et al, Robust Stability of NCS with Time-varying Network-induced Delays," CDC,2006

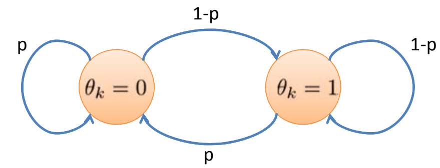
Control designs that are robust to packet dropouts[3]

$$z_{k+1} = \Phi_\theta z_k$$

where

$$\Phi_\theta = \begin{bmatrix} e^{Ah} + \theta\Gamma(h-\tau)BC & e^{A(h-\tau)}\Gamma(\tau)B + (1-\theta)\Gamma(h-\tau)B \\ \theta C & (1-\theta)I \end{bmatrix}$$

for $\theta \in 0, 1$



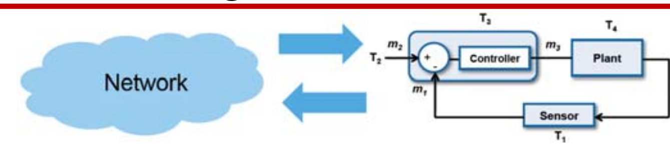
Theorem:
The NCS given above with dropout probability p (Bernoulli) is exponentially mean-square stable if there exists a symmetric matrix $Z > 0$ such that

$$\begin{bmatrix} Z & \sqrt{p}(\Phi_0 Z)' & \sqrt{1-p}(\Phi_1 Z)' \\ * & Z & 0 \\ * & * & Z \end{bmatrix} > 0$$

(3) P. Seiler and R. Sengupta, "Analysis of communication losses in vehicle control problems," Amer. Contr. Conf., 2001, vol. 2

ARBITRATED NETWORKED CONTROL SYSTEMS (ANCS)

ANCS: Co-design Network and Control



NCS: Given a network, how do we design the controller?

ANCS:

- Exploit network transparency. Use information available
- Exploit network flexibility. Given a controller, how do we design the network?
- Co-design the network and controller.

ANCS:
Co-design τ to meet quality of control and network resource constraints.

A Standard Sampled-Data Problem

- Physical system: $\dot{x}(t) = A_c x(t) + B_c u(t)$
- Multiple both sides by $e^{-A_c t}$

$$\frac{d}{dt}(e^{-A_c t}x(t)) = e^{-A_c t}B_c u(t)$$

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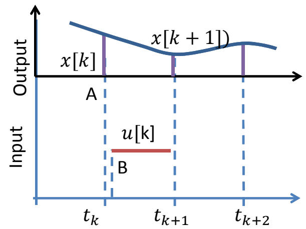

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Effect of Network: $\tau \neq 0$

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$$x[k+1] = e^{A_c h}x[k] + \int_{kh}^{kh+h} e^{-A_c(\eta-kh-h)}B_c\,u(\eta)d\eta$$

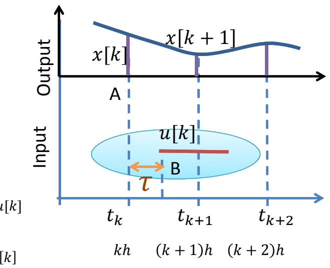
- $u(\eta) = u[k-1], \quad [\mathrm{t_k}, \mathrm{t}_k + \tau]$
- $u(\eta) = u[k], \quad [\mathrm{t}_k + \tau, t_{k+1}]$

$$x[k+1] = e^{A_c h}x[k] +$$

$$\int_{kh}^{kh+\tau} e^{-A_c(\eta-kh-h)}B_c\,d\eta \cdot u[k-1] + \int_{kh+\tau}^{kh+h} e^{-A_c(\eta-kh-h)}B_c\,d\eta \cdot u[k]$$

$$= \underbrace{e^{A_c h}}_{A}x[k] + \underbrace{\int_{h-\tau}^{h} e^{A_c \nu}\,d\eta \cdot B_c}_{B_2} \cdot u[k-1] + \underbrace{\int_{0}^{h-\tau} e^{A_c \nu}\,d\eta \cdot B_c}_{B_1} \cdot u[k]$$

$$\Rightarrow x[k+1] = Ax[k] + B_1 u[k] + B_2 u[k-1]$$



- τ can be a significant fraction of h.
- Design $u[k]$ using A, B_1, and B_2 and the size of τ in relation to h.

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The overall idea

- Plant-model:

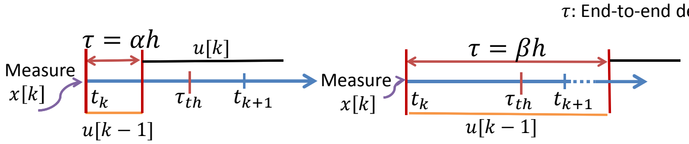
$$\dot{x}(t) = A_c x(t) + B_c u(t-\tau) \qquad \text{(continuous-time)}$$

- Sample at t_k and t_{k+1}:

$$x[k+1] = Ax[k] + B_1 u[k] + B_2 u[k-1]$$

- At each t_k: Measure $x[k]$, compute $u[k]$after τ

τ: End-to-end delay



Nominal: $\alpha < 1, \alpha \leq \tau_{th}$ Drop: β large, $\tau > \tau_{th}$

- $\tau = \begin{cases} \alpha h, \\ \beta h, \end{cases}$ - depending on the applications serviced

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ANCS TOOLS

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Stability Tools

We need to accommodate different cases where $\tau \sim 0, \tau \sim \alpha h, \alpha \ll 1, \tau \sim \beta h, \beta \sim 1$.
- Switched Systems

- Dwell time
- Common Lyapunov Function
- Multiple Lyapunov Functions (MLF)
- Definition:

  A is Schur $\Rightarrow$ all eigenvalues of A are inside the unit circle

Dwell time

- Switch between

$$\begin{cases} x_{k+1} = A_n x_k, & If \quad Nominal \\ x_{k+1} = A_d x_k, & If \quad Drop \end{cases}$$

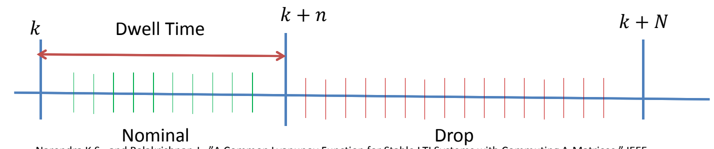
- A_n: Schur; A_d: arbitrary

$$\|x_{k+N}\| \leq \|A_n^n\| \quad \|A_d\|^{N-n} \cdot \|x_0\|$$

- Make n large compared to $N-n$

$$\|x_{k+N}\| \leq \gamma \lambda^n k^{N-n} \cdot \|x_0\|, \quad \lambda < 1$$

- For some n, $\gamma \lambda^n k^{N-n} < 1; \Rightarrow$ stability of the switched system



Narendra K.S., and Balakrishnan J., "A Common Lyapunov Function for Stable LTI Systems with Commuting A-Matrices," IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 39. NO. 12, DECEMBER 1994

Common Lyapunov Function (CLF)

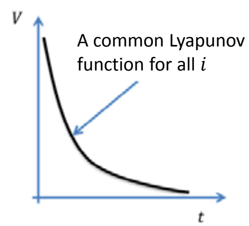
- Hurwitz Matrices A_i
- $\dot{x} = A_i x, \quad i = 1,2,\cdots,N$
  - Stable with arbitrary switching* if for any $i, j \in \sigma$

$$A_i A_j = A_j A_i$$

- A more powerful tool: CLF

$$\mathrm{V} = x^T P x$$

$$A_i^T P + A_i P = -Q \qquad Q \succ 0$$



* Narendra K.S., and Balakrishnan J., "A Common Lyapunov Function for Stable LTI Systems with Commuting A-Matrices," IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 39. NO. 12, DECEMBER 1994

Common Quadratic Lyapunov Function (CQLF)

- Discrete-time Systems: $x[k+1] = A_i x[k]$
  - A_i Schur $\quad i = 1, \ldots, n$
  - Switched system is stable with arbitrary switching

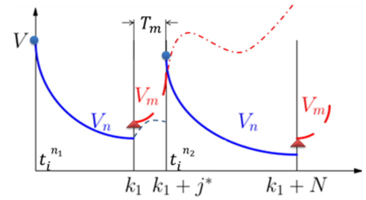
    if there exist $P \succ 0$ such that

$$V = x^T P x \qquad A_i^T P A_i - P < 0$$

Narendra K.S., and Balakrishnan J., "A Common Lyapunov Function for Stable LTI Systems with Commuting A-Matrices," IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 39. NO. 12, DECEMBER 1994

Multiple Lyapunov Functions

Definition (LLF):
- $V_i(x)$ is a Lyapunov-Like Function (LLF) if
  - $V_i(x) > 0$
  - $V_i(x[k_1]) \leq h(V_i(x[k_1 + j_1^*]))$
- h: continuous; $h(0) = 0$.



MLF Theorem: $x[k+1] = A_i x[k]$. Switched system is stable if

(i) LLFs V_is exist over all intervals T_i's where ith system is active.

(ii) For all switching instants t_i^j,

$$V_i(x[t_i^{n_2}]) \leq V_i(x[t_i^{n_1}]);$$

1) Soudbakhsh D., Phan L.X, Sokolsky O., Lee I., and Annaswamy A.M., "Co-design of control and platform with dropped signals," ICCPS 2013.
2) Branicky M.S., "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," IEEE-TAC, 43(4):475 – 482, 1998.

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Linear Matrix Inequalities (LMI)

- LMI in the variable $x \in \Re^n$ is an inequality:

$$a(x) = a_0 + x_1 a_1 + \cdots + x_n a_n \succcurlyeq 0$$

where $a_0, a_1, \cdots, a_n$ are symmetric $m \times m$ matrices

- Can be solved for x very efficiently
- Example: Lyapunov Inequality $A^T P + PA \prec 0$ is an LMI in variable P

Boyd S., El-Ghaoui L., Feron E., and Balakrishnan V., *"Linear matrix inequalities in system and control theory,"* Vol. 15. Philadelphia: Society for Industrial and Applied Mathematics, 1994.

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Summary (Part 2)

- Control theory fundamentals
  - Use of Feedback
  - Control performance metrics
    - Transient and steady-state
    - Trade-off between speed/accuracy and control effort
- ANCS
  - A Network Control System that exploits the information available and flexibility in the platform design
  - Transparency and flexibility in the network: Delays are known
  - Use of switching systems and their design for stability

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